

On mysteriously missing T-duals, H-flux and the T-duality group

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A general formula for the topology and H-flux of the T-duals of type II string theories with H-flux on toroidal compactifications is presented here. It is known that toroidal compactifications with H-flux do not necessarily have T-duals which are themselves toroidal compactifications. A big puzzle has been to explain these mysterious “missing T-duals”, and our paper presents a solution to this problem using noncommutative topology. We also analyze the T-duality group and its action, and illustrate these concepts with examples.

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T-duality is a symmetry of type II string theories that involves exchanging a theory compactified on a torus with a theory compactified on the dual torus. The T-dual of a type II string theory compactified on a circle, in the presence of a topologically nontrivial NS 3-form H-flux, was analyzed in special cases in [2, 5, 6, 7]. There it was observed that T-duality changes not only the H-flux, but also the spacetime topology. A general formalism for dealing with T-duality for compactifications arising from a free circle action was developed in [8]. This formalism was shown to be compatible with two physical constraints: (1) it respects the local Buscher rules [1], and (2) it yields an isomorphism on twisted K-theory, in which the Ramond-Ramond charges and fields take their values [11, 12, 13]. It was shown in [8] that T-duality exchanges the first Chern class with the fiberwise integral of the H-flux, thus giving a formula for the T-dual spacetime topology. In this note we will present an account for physicists of the results in [16], consisting of a formula for the T-dual of a toroidal compactification, that is a theory compactified via a free torus action, with H-flux. One striking new feature that occurs for higher dimensional tori is that not every toroidal compactification with H-flux has a T-dual; moreover, even if it has a T-dual, then the T-dual need not be another toroidal compactification with H-flux. A big puzzle has been to explain these mysterious “missing T-duals”, and our paper presents a solution to this problem using noncommutative topology. A similar phenomenon was noticed in [15] in the special case of the trivial \mathbb{T}^2 bundle over \mathbb{T} with non-trivial H-flux. We also show that the generalized T-duality group $GO(n, n; \mathbb{Z})$, n being the rank of the torus, acts to generate the complete list of T-dual pairs related to a given toroidal compactification with H-flux. We will explain these results by providing examples and applications.

In this letter we will consider type II string theories on target d -dimensional manifolds X , which are assumed to admit free, rank n torus actions. While for most physical applications one wants $d = 10$, we do not need to assume this, and in fact X could represent a partial reduction of the original 10-dimensional spacetime after preliminary compactification in $10-d$ dimensions. The space of orbits of the torus action on X is given by a $(d-n)$ -dimensional

manifold, which we call Z . The freeness of the action implies that each orbit is a torus and that none of these tori degenerate. As a result X is a principal torus bundle over the base Z , and so its topology is entirely determined by the topology of the base Z together with the first Chern class c of the bundle $X \xrightarrow{p} Z$ in $H^2(Z, \mathbb{Z})$. This viewpoint is useful in that it automatically identifies some gauge equivalent configurations, excludes configurations not satisfying some equations of motion and imposes the Dirac quantization conditions. The Chern class c is represented by a vector valued closed 2-form with integral periods, the curvature F . We will discuss conditions under which the pair $(X \xrightarrow{p} Z, H)$ has a T-dual, either another pair $(X \xrightarrow{p^\#} Z, H^\#)$ with the same base Z (the “classical” case) or a more general non-commutative object (the “nonclassical” case). In both cases, there should be a sense in which string theory on the original space X (with H-flux H) is equivalent to a theory on the T-dual.

Basic setup: Let $p: X \rightarrow Z$ be a principal T -bundle as above, where $T = (S^1)^n = \mathbb{T}^n$ is a rank n torus. Let $H \in H^3(X, \mathbb{Z})$ be an H-flux on X satisfying $\iota^*H = 0$, $\iota^*: H^3(X, \mathbb{Z}) \rightarrow H^3(T, \mathbb{Z})$, where $\iota: T \hookrightarrow X$ is the inclusion of a fiber. (This condition is automatically satisfied when $n \leq 2$.)

The simplest case when the condition $\iota^*(H) = 0$ does not apply is $X = \mathbb{T}^3$, when considered as a rank 3, principal torus bundle over a point, with H-flux a non-zero integer multiple of the volume 3-form on \mathbb{T}^3 . When $\iota^*(H) \neq 0$, there is no T-dual in the sense we are considering, even in what we call the “nonclassical” sense.

It turns out that nontrivial bundles are always T-dual to trivial bundles with non-zero H-flux. Therefore we will need to include the fluxes H and $H^\#$ in our toroidal compactifications, which are then topologically determined by the triples (Z, c, H) and $(Z, c^\#, H^\#)$, where H and $H^\#$ are closed three-forms on the total spaces X and $X^\#$ respectively.

Our result on classical T-duals: Suppose that we are in the basic setup as above. Choose a basis $\{\mathbb{T}_j^2\}_{j=1}^k$, $k = \binom{n}{2}$ for $H_2(T, \mathbb{Z})$ consisting of 2-tori, and push this forward into $H_2(X, \mathbb{Z})$ via ι_* . We

can consider the cohomology classes

$$\int_{\mathbb{T}_j^2} H = H \cap \iota_*(\mathbb{T}_j^2) \in H^1(X, \mathbb{Z}).$$

These classes restrict to 0 on the fibers, since $\iota^*(H) = 0$. Using the following exact sequence, derived from the spectral sequence of the torus bundle,

$$0 \rightarrow H^1(Z, \mathbb{Z}) \xrightarrow{p^*} H^1(X, \mathbb{Z}) \xrightarrow{\iota^*} H^1(T, \mathbb{Z}) \rightarrow \dots, \quad (1)$$

we see that the classes $\int_{\mathbb{T}_j^2} H = H \cap \iota_*(\mathbb{T}_j^2) \in H^1(X, \mathbb{Z})$ come from unique classes $\{\beta_j\}_{j=1}^k$ in $H^1(Z, \mathbb{Z})$. Set

$$p_!(H) = (\beta_1, \dots, \beta_k) \in H^1(Z, \mathbb{Z}^k). \quad (2)$$

If $p_!(H) = 0 \in H^1(Z, \mathbb{Z}^k)$, and in particular if Z is simply connected, then there is a classical T-dual to (p, H) , consisting of $p^\# : X^\# \rightarrow Z$, which is another principal T -bundle over Z , and $H^\# \in H^3(X^\#, \mathbb{Z})$, the T-dual H-flux on $X^\#$. One obtains a commuting diagram of the form

$$\begin{array}{ccc} & X \times_Z X^\# & \\ p^*(p^\#) \swarrow & & \searrow (p^\#)^*(p) \\ X & & X^\# \\ p \searrow & & \swarrow p^\# \\ & Z & \end{array} \quad (3)$$

In this case, the compactifications topologically specified by (Z, c, H) and $(Z, c^\#, H^\#)$ are T-dual if $c, c^\# \in H^2(Z, \mathbb{Z}^n)$ are related as follows:

Let $c_j, j = 1, \dots, n$, be the components of c . Let $X_j \xrightarrow{\pi_j} Z$ be the principal \mathbb{T}^{n-1} subbundle of X obtained by deleting c_j , i.e. the Chern class of X_j is

$$c(\pi_j) = (c_1, \dots, \hat{c}_j, \dots, c_n).$$

Then $X \xrightarrow{p_j} X_j$ is a principal S^1 -bundle whose Chern class is equal to $\pi_j^*(c_j)$. Define $X_j^\# \xrightarrow{\pi_j^\#} Z$, $X^\# \xrightarrow{p_j^\#} X_j^\#$ etc. similarly. Then we have

$$(\pi_j)^*(c_j^\#) = (p_j)_!(H) \quad \text{and} \quad (\pi_j^\#)^*(c_j) = (p_j^\#)_!(H^\#).$$

Here the correspondence space $X \times_Z X^\#$ is the submanifold of $X \times X^\#$ consisting of pairs of points (x, y) such that $p(x) = p^\#(y)$, and has the property that it implements the T-duality between (p, H) and $(p^\#, H^\#)$. It also turns out that $p_!^\#(H^\#) = 0 \in H^1(Z, \mathbb{Z}^k)$ and that

the T-dual of $(p^\#, H^\#)$ is (p, H) . So in this case, T-duality exchanges the integral of the H-flux (over a basis of circles in the fibers) with the first Chern class. The condition in the result above determines, at the level of cohomology, the curvatures F and $F^\#$. However the NS field strengths are only determined up to the addition of a three-form on the base Z , because the integral of such a form over a basis of circles in the fibers vanishes. This settles a conjecture in [8], and was also considered by [9].

The simplest higher rank example is $X = S^2 \times \mathbb{T}^2$, considered as the trivial \mathbb{T}^2 bundle over $Z = S^2$, with H-flux equal to $H = k_1 a \wedge b_1 + k_2 a \wedge b_2$, where we use the Künneth theorem to identify $H^3(S^2 \times \mathbb{T}^2, \mathbb{Z})$ with $H^2(S^2, \mathbb{Z}) \otimes H^1(\mathbb{T}^2, \mathbb{Z})$, and a is the generator of $H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$, b_1, b_2 are the generators of $H^1(\mathbb{T}^2, \mathbb{Z}) \cong \mathbb{Z}^2$ and $k_1, k_2 \in \mathbb{Z}$. Since S^2 is simply connected, $p_!(H) = 0$ and the T-dual of $(S^2 \times \mathbb{T}^2, H)$ is the nontrivial rank 2 torus bundle P over S^2 with Chern class $c_1(P) = (k_1 a, k_2 a) \in H^2(S^2, \mathbb{Z}) \oplus H^2(S^2, \mathbb{Z}) = H^2(S^2, \mathbb{Z}^2)$, and with H-flux equal to zero. This example generalizes easily by taking the Cartesian product with a manifold M , and pulling back the H-flux to the product and arguing as before, we see that the T-dual of $(M \times S^2 \times \mathbb{T}^2, H)$ is $(M \times P, 0)$.

Our result on nonclassical T-duals: Suppose that we are in the basic setup as above. If $p_!(H) \neq 0 \in H^1(Z, \mathbb{Z}^k)$, then there is no classical T-dual to (p, H) ; however, there is a nonclassical T-dual consisting of a continuous field of (stabilized) noncommutative tori A_f over Z , where the fiber over the point $z \in Z$ is equal to the rank n noncommutative torus $A_{f(z)}$ (see Figure 1 below). Here $f : Z \rightarrow \mathbb{T}^k$ is a continuous map representing $p_!(H)$.

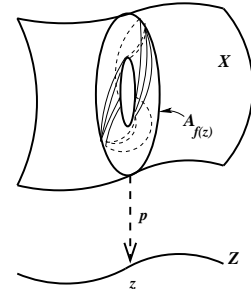


FIG. 1: In the diagram, the fiber over $z \in Z$ is the noncommutative torus $A_{f(z)}$, which is represented by a foliated torus, with foliation angle equal to $f(z)$.

This suggests an unexpected link between classical string theories and the “noncommutative” ones, obtained by “compactifying” matrix theory on tori, as in [4] (cf. also [19, §§6–7]). We now recall the definition of the rank n noncommutative torus A_θ , cf. [18]. This algebra (stabilized by tensoring with the compact operators \mathcal{K}) occurs geometrically as the foliation algebra associated to Kromer foliations on the torus [3]. In [4], the same algebra

occurs naturally from studying the field equations of the IKKT (Ishibashi-Kawai-Kitazawa-Tsuchiya) model compactified on n -tori, or from the study of BPS states of the BFSS (Banks-Fisher-Shenker-Susskind) model. (The IKKT and BFSS models are both large- N matrix models in which Poisson brackets in the Lagrangian are replaced by matrix commutators.) For each $\theta \in \mathbb{T}^k$, identified with a hermitian matrix $\theta = (\theta_{ij})$, $i, j = 1, \dots, n$, $\theta_{ij} \in S^1$ with 1's down the diagonal, the *noncommutative torus* A_θ is defined abstractly as the C^* -algebra generated by n unitaries U_j , $j = 1, \dots, n$ in an infinite dimensional Hilbert space satisfying the commutation relation $U_i U_j = \theta_{ij} U_j U_i$, $i, j = 1, \dots, n$. Elements in A_θ can be represented by infinite power series

$$f = \sum_{m \in \mathbb{Z}^n} a_m U^m, \quad (4)$$

where $a_m \in \mathbb{C}$ and $U^m = U_1^{m_1} \dots U_n^{m_n}$, for all $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$.

A famous example of a principal torus bundle with non-T-dualizable H-flux is provided by \mathbb{T}^3 , considered as the trivial \mathbb{T}^2 -bundle over \mathbb{T} , with H given by k times the volume form on \mathbb{T}^3 . H is non T-dualizable in the classical sense since $p_1(H) \neq 0 \in H^1(\mathbb{T}, \mathbb{Z})$. Alternatively, there are no non-trivial principal \mathbb{T}^2 -bundles over \mathbb{T} , since $H^2(\mathbb{T}, \mathbb{Z}^2) = 0$, that is, there is no way to dualize the H-flux by a (principal) torus bundle over \mathbb{T} , cf. [7]. This is an example of a mysteriously missing T-dual. This example is covered by our result on nonclassical T-duals above. The T-dual is realized by a field of stabilized *noncommutative tori* fibered over \mathbb{T} . Let $\mathcal{H} = L^2(\mathbb{T})$ and consider the projective unitary representation $\rho_\theta: \mathbb{Z}^2 \rightarrow \text{PU}(\mathcal{H})$ in which the generator of the first \mathbb{Z} factor acts by multiplication by z^k (where \mathbb{T} is thought of as the unit circle in \mathbb{C}) and the generator of the second \mathbb{Z} factor acts by translation by $\theta \in \mathbb{T}$. Then the Mackey obstruction of ρ_θ is $\theta^k \in \mathbb{T} \cong H^2(\mathbb{Z}^2, \mathbb{T})$. Let $\mathcal{K}(\mathcal{H})$ denote the algebra of compact operators on \mathcal{H} and define an action α of \mathbb{Z}^2 on continuous functions on the circle with values in compact operators, $C(\mathbb{T}, \mathcal{K}(\mathcal{H}))$, given at the point θ by ρ_θ . Define the C^* -algebra B , which is obtained by inducing the \mathbb{Z}^2 action to an action of \mathbb{R}^2 on $B = \text{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2}(C(\mathbb{T}, \mathcal{K}(\mathcal{H})), \alpha)$, i.e. $B = \{f: \mathbb{R}^2 \rightarrow C(\mathbb{T}, \mathcal{K}(\mathcal{H})) : f(t+g) = \alpha(g)(f(t)), t \in \mathbb{R}^2, g \in \mathbb{Z}^2\}$. Then B is a continuous-trace C^* -algebra having spectrum \mathbb{T}^3 and Dixmier-Douady invariant H . B also has an action of \mathbb{R}^2 whose induced action on the spectrum of B is the trivial bundle $\mathbb{T}^3 \rightarrow \mathbb{T}$. Then our noncommutative T-dual is the crossed product algebra $B \rtimes \mathbb{R}^2 \cong C(\mathbb{T}, \mathcal{K}(\mathcal{H})) \rtimes \mathbb{Z}^2 = A_f$, which has fiber over $\theta \in \mathbb{T}$ given by $\mathcal{K}(\mathcal{H}) \rtimes_{\rho_\theta} \mathbb{Z}^2 \cong A_\theta \otimes \mathcal{K}(\mathcal{H})$, where A_θ is the noncommutative 2-torus. In fact, the crossed product $B \rtimes \mathbb{R}^2$ is isomorphic to the (stabilized) group C^* -algebra $C^*(H_\mathbb{Z}) \otimes \mathcal{K}$, where $H_\mathbb{Z}$ is the integer Heisenberg-type

group,

$$H_\mathbb{Z} = \left\{ \begin{pmatrix} 1 & x & \frac{1}{k}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}. \quad (5)$$

In summary, the nonclassical T-dual of $(\mathbb{T}^3, H = k)$ is $A_f = C^*(H_\mathbb{Z}) \otimes \mathcal{K}$. As required in order to match up RR charges, the K -theory of this algebra is the same as the K -theory of \mathbb{T}^3 with twist given by our H-flux, or k times the volume form.

This example generalizes easily by taking the Cartesian product with a manifold M . Pulling back the H -flux to the product and arguing as before, we see that $(M \times \mathbb{T}^3, H = k)$ is T-dual to $C(M) \otimes C^*(H_\mathbb{Z}) \otimes \mathcal{K}$. For instance, if the dimension of M is seven, then $M \times \mathbb{T}^3$ is ten dimensional, yielding examples of spacetime manifolds that are relevant to type II string theory.

Our results on the T-duality group:

It is important to realize that a fixed space X can sometimes be given the structure of a principal torus bundle over Z in many different ways. For example, given a free action of a torus $T = \mathbb{T}^n$ on X , with quotient space $Z = X/T$, we can for every element $g \in \text{Aut}(\mathbb{T}^n) = GL(n, \mathbb{Z})$ define a new free action of T on X , twisted by g , by the formula $x \cdot_g t = x \cdot g(t)$. (Here $t \in T$, \cdot is the original free right action of T on X , and \cdot_g is the new twisted action.) If $c \in H^2(Z, \mathbb{Z}^n)$ was the Chern class of the original bundle, the Chern class of the g -twisted bundle is $g \cdot c$, with g acting via the action of $GL(n, \mathbb{Z})$ on \mathbb{Z}^n .

The group $GL(n, \mathbb{Z})$ embeds in $O(n, n; \mathbb{Z})$, the subgroup of $GL(2n, \mathbb{Z})$ preserving the quadratic form defined by $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, via $a \mapsto \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix}$ (see [10, §2.4]). This larger group $O(n, n; \mathbb{Z})$ is often called the *T-duality group*. In fact we will consider the still larger *generalized T-duality group* $GO(n, n; \mathbb{Z}) = O(n, n; \mathbb{Z}) \rtimes (\mathbb{Z}/2)$ of matrices in $GL(2n, \mathbb{Z})$ preserving the form $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ up to sign. Good references for the T-duality group include [10] (for the state of the theory up to 1994) and [14] for more current developments.

Suppose that we are in the basic setup as above, with Z simply connected, so that one is always guaranteed to have a classical T-dual. Then the generalized T-duality group $GO(n, n; \mathbb{Z})$ acts on the set of T-dual pairs (p, H) and $(p^\#, H^\#)$ to generate all related T-dual pairs. All of these pairs are physically equivalent. The restriction of the action to $GL(n, \mathbb{Z})$ (as embedded above) corresponds to twisting of the action on the same underlying space as above.

When Z is not simply connected and $p_1(H) \neq 0$, it is not clear that one has an action of the full T-duality group. But the action of $GL(n, \mathbb{Z})$ always sends the pair consisting of (p, H) and its nonclassical T-dual to another nonclassical pair, involv-

ing continuous fields of (stabilized) noncommutative tori over Z .

We illustrate the action of the generalized T-duality group in the simplest case of circle bundles with H-flux, in which case the generalized T-duality group reduces to $GO(1, 1; \mathbb{Z})$, a dihedral group of order 8.

Consider the example of the 3 dimensional lens space $L(1, p) = S^3/\mathbb{Z}_p$, with H-flux $H = q$ times the volume form, cf. [17]. Here $p, q \in \mathbb{Z}$, and initially we take $p, q > 0$. Then $L(1, p)$ is a circle bundle over the 2-dimensional sphere S^2 and has first Chern class equal to p times the volume form of S^2 . Then, as shown in [8, §4.3], $(L(1, p), H = q)$ and $(L(1, q), H = p)$ are T-dual to each other, and the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of $O(1, 1; \mathbb{Z})$ interchanges them. The element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of the T-duality group $O(1, 1; \mathbb{Z})$ lies in the subgroup $GL(1, \mathbb{Z})$, embedded as above, and acts by twisting the S^1 action on $L(1, p)$. This twisted action makes $L(1, p)$ into a circle bundle over S^2 having first Chern class equal to $-p$ times the volume form of S^2 . This bundle is denoted $L(1, -p)$, and its total space is diffeomorphic to $L(1, p)$, though by an orientation-reversing diffeomorphism. Therefore the action of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ on the pair $(L(1, p), H = q)$ and $(L(1, q), H = p)$ gives rise to a new T-dual pair $(L(1, -p), H = -q)$ and $(L(1, -q), H = -p)$. The group $GO(1, 1; \mathbb{Z})$ is generated by the two elements

of $O(1, 1; \mathbb{Z})$ just discussed and by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which replaces the original T-dual pair by the pair consisting of $(L(1, p), H = -q)$ and $(L(1, -q), H = p)$. Here we have tacitly assumed $p, q \geq 2$; we can extend things to other values of p and q by making the convention that $L(1, 1) = S^3$ and $L(1, 0) = S^2 \times S^1$. This refines the T-duality in [8, §4.3]. Thus in general there are 8 different (bundle, H-flux) pairs with equivalent physics, corresponding to $(\pm p, \pm q)$ and $(\pm q, \pm p)$.

This example generalizes easily by taking the Cartesian product with a manifold M . For instance, if the dimension of M is seven, then we obtain 8 different (bundle, H-flux) pairs in the same $GO(1, 1; \mathbb{Z})$ -orbit as $M \times L(1, p)$. All of these are ten-dimensional spacetime manifolds relevant to type II string theory.

We end with some open problems. A critical verification of any proposed duality is that the anomalies should match on both sides. This was checked for T-duality involving circle bundles with H-flux in [8], but remains to be analyzed in the general torus bundle case with H-flux. It also remains to be determined whether or not the group $GO(n, n; \mathbb{Z})$ also operates in the nonclassical case. Another problem is to extend our results to non-free torus actions [20], in which case it could be relevant to mirror symmetry.

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- [1] T. Buscher, Phys. Lett. **B194**, 59 (1987); *ibid.*, Phys. Lett. **B201**, 466 (1988).
 - [2] E. Álvarez, L. Álvarez-Gaumé, J. L. F. Barbón and Y. Lozano, Nucl. Phys. **B415**, 71 (1994), hep-th/9309039.
 - [3] A. Connes, in *Operator algebras and applications, Part I* (Kingston, Ont., 1980), R. V. Kadison, ed., pp. 521–628, Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, R.I., 1982.
 - [4] A. Connes, M. R. Douglas and A. Schwarz, J. High Energy Phys. 02 (1998) 003, hep-th/9711162.
 - [5] M. J. Duff, H. Lü and C. N. Pope, Nucl. Phys. **B532**, 181 (1998), hep-th/9803061.
 - [6] S. Gurrieri, J. Louis, A. Micu and D. Waldram, Nucl. Phys. **B654**, 61 (2003), hep-th/0211102.
 - [7] S. Kachru, M. Schulz, P. Tripathy and S. Trivedi, J. High Energy Phys. 03 (2003) 061, hep-th/0211182.
 - [8] P. Bouwknegt, J. Evslin and V. Mathai, Comm. Math. Phys. **249**, 383 (2004), hep-th/0306062; *ibid.*, Phys. Rev. Lett. **92**, 181601 (2004).
 - [9] P. Bouwknegt, K. Hannabuss and V. Mathai, J. High Energy Phys. 03 (2004) 018, hep-th/0312284.
 - [10] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. **244**, no. 2–3, 77 (1994), hep-th/9401139.
 - [11] R. Minasian and G. Moore, J. High Energy Phys. 11 (1997) 002, hep-th/9710230.
 - [12] E. Witten, J. High Energy Phys. 12 (1998) 019, hep-th/9810188.
 - [13] P. Bouwknegt and V. Mathai, J. High Energy Phys. 03 (2000) 007, hep-th/0002023.
 - [14] C. M. Hull, hep-th/0406102.
 - [15] D. A. Lowe, H. Nastase and S. Ramgoolam, Nucl. Phys. **B667** 55 (2003), hep-th/0303173.
 - [16] V. Mathai and J. Rosenberg, Comm. Math. Phys., **253** 705 (2005), hep-th/0401168; *ibid.*, work in progress.
 - [17] J. Maldacena, G. Moore and N. Seiberg, J. High Energy Phys. 10 (2001) 005, hep-th/0108152.
 - [18] M. Rieffel, Pacific J. Math. **93**, 415 (1981).
 - [19] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032, hep-th/9908142.
 - [20] R. Donagi and T. Pantev, math.AG/0306213.